United Kingdom
Mathematics Trust

Intermediate Mathematical Olympiad

Maclaurin paper

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Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions.

It is not intended that these solutions should be thought of as the 'best' possible solutions and the ideas of readers may be equally meritorious.

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1. A bag contains counters, of which ten are coloured blue and Y are coloured yellow. Two yellow counters and some more blue counters are then added to the bag. The proportion of yellow counters in the bag remains unchanged before and after the additional counters are placed into the bag.

Find all possible values of Y.

SOLUTION

Let X be the number of blue counters added to the bag. The proportion of yellow counters in the bag at the beginning is

$$\frac{Y}{Y+10}$$

and the proportion of yellow counters in the bag at the end is

$$\frac{Y+2}{X+Y+12}$$

Hence we have

$$\frac{Y}{10+Y}=\frac{Y+2}{X+Y+12}$$

Cross-multiplying, we obtain $Y(X+Y+12)=(Y+2)(10+Y)$, which gives

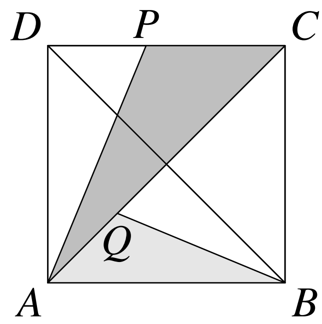
$$XY+Y^2+12Y=10Y+Y^2+20+2Y$$

This simplifies to $XY=20$. Hence the number of yellow counters originally in the bag is either 1, 2, 4, 5, 10 or 20.

(Two other valid equations, $\frac{Y}{10}=\frac{Y+2}{X+10}$ and $\frac{Y}{10}=\frac{2}{X}$, also lead to $XY=20$)

2. In the square $ABCD$, the bisector of $\angle CAD$ meets CD at P and the bisector of $\angle ABD$ meets AC at Q.

What is the ratio of the area of triangle ACP to the area of triangle BQA?



SOLUTION

We know that, since AC is the diagonal of a square, $\angle PCA = 45^\circ = \angle QAB$. Also $\angle DAC = 45^\circ = \angle DBA$. Since AP and BQ are angle bisectors, $\angle PAC = 22.5^\circ = \angle QBA$.

Hence the triangles CPA and AQB have equal angles, so they are similar. Now $CA = \sqrt{2}$ and $AB = 1$, so the sides are in the ratio $\sqrt{2} : 1$, and the areas are in the ratio $2 : 1$.

3. An altitude of a triangle is the shortest distance from a vertex to the line containing the opposite side.

Find the side lengths of all possible right-angled triangles with perimeter 5 cm and shortest altitude 1 cm.

Solution

The three altitudes of a right-angled triangle are the two sides around the right angle and the perpendicular from the right angle to the hypotenuse, which is clearly the shortest.

Let the lengths of the three sides in cm be a, b, c, where c is the hypotenuse.

(a) From the perimeter, we know that $a + b + c = 5$.

(b) The area using base a and height b is $\frac{1}{2}ab$ and the area using base c and height 1 is $\frac{1}{2}c$, so we obtain $ab = c$.

(c) From Pythagoras, we have $a^2 + b^2 = c^2$.

From (a) we have $a + b = 5 - c$, which, when squared, gives $a^2 + 2ab + b^2 = 25 - 10c + c^2$.

Substituting from (b) and (c), this becomes $c^2 + 2c = 25 - 10c + c^2$, so $12c = 25$ and $c = \frac{25}{12}$.

Now we know that $a + b = \dfrac{35}{12}$ and $ab = \dfrac{25}{12}$.

Hence $a + \dfrac{25}{12a} = \dfrac{35}{12}$ and so $12a^2 - 35a + 25 = 0$, which has solutions $a = \frac{5}{3}$ or $\frac{5}{4}$.

Finally the corresponding values of b are $\frac{5}{4}$ and $\frac{5}{3}$. Hence there is a unique triangle with sides $\frac{5}{3}$ cm, $\frac{5}{4}$ cm and $\frac{25}{12}$ cm.

An alternative approach uses $a + b + ab = 5$ to obtain $a = \dfrac{5 - b}{1 + b}$.

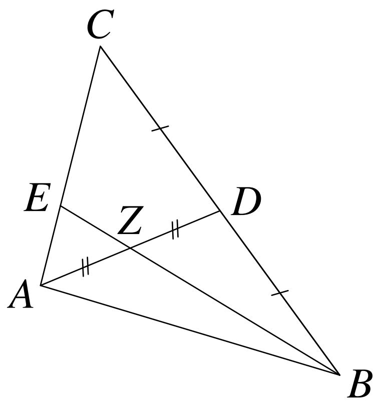
Hence $(\frac{5-b}{1+b})^2(b^2 - 1) = b^2$, so we have $(5 - b)^2(b - 1) = b^2(1 + b)$ and this, when expanded, gives $12b^2 - 35b + 25 = 0$ and so $(3b - 5)(4b - 5) = 0$.

Hence $b = \frac{5}{3}$ or $\frac{5}{4}$ and we find a and c as before.

Since this argument is not obviously reversible, it is necessary to check that these lengths satisfy the requirements of the problem.

4. The diagram shows a triangle ABC and two lines AD and BE, where D is the midpoint of BC and E lies on CA. The lines AD and BE meet at Z, the midpoint of AD.

What is the ratio of the length CE to the length EA?

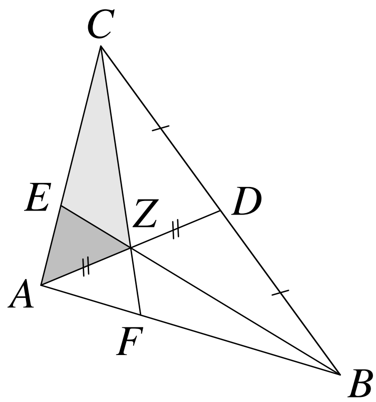


SOLUTION

A **Using areas**

Since $CD = DB$, $[CDZ] = [BDZ]$ and, since $DZ = ZA$, $[BDZ] = [BAZ]$.

Now, considering triangles on the side CA, we have $\frac{[CEZ]}{[AEZ]} = \frac{[CZB]}{[AZB]} = 2$, and it follows that $CE : EA = 2 : 1$.



What is being used here is the useful ‘arrowhead’ shortcut which links the shaded areas directly to the unshaded ones. It is more likely that solvers will proceed via the step

$\frac{[CEZ]}{[AEZ]} = \frac{[CEB]}{[AEB]}$ and then use a bit of algebra. Alternatively, some might be aware that if $\frac{a}{b} = \frac{c}{d}$, it follows that $\frac{a}{b} = \frac{a-c}{b-d}$.

B **Using vectors**

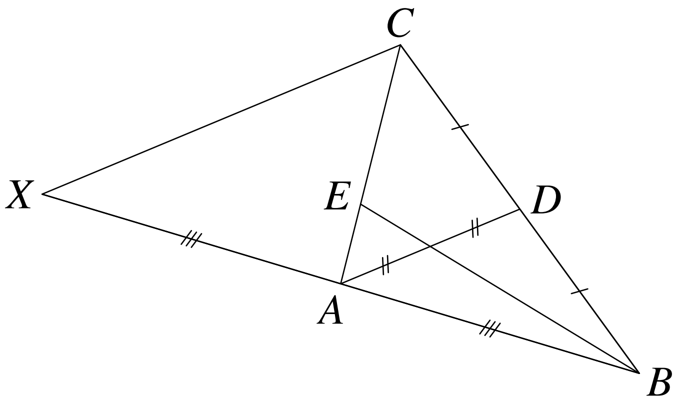
Let A be the origin and denote B and C by $4\mathbf{b}$ and $4\mathbf{c}$ respectively. Then D is $2(\mathbf{b} + \mathbf{c})$ and Z is $\mathbf{b} + \mathbf{c}$. Now E lies on both BZ and AC, so there are real numbers λ, μ such that $4\mu\mathbf{c} = \lambda(\mathbf{b} + \mathbf{c}) + 4(1 - \lambda)\mathbf{b}$. Hence $\lambda = \frac{4}{3}$ and $\mu = \frac{1}{3}$, so $CE : AE = 2 : 1$.

C **Using construction**

Let X be on BA such that $BA = AX$. Then CA is a median of triangle BCX.

The triangles BDA and BCX are similar, by ratios, so BE, which is a median of BDA, is also a median of BCX.

Hence E is the centroid of BCX and so $CE : EA = 2 : 1$



There are several other constructions which work. Other approaches use coordinate geometry, vectors and complex numbers. Ratio theorems such as Ceva and Menelaus are also useful.

5. Let p and q respectively be the smallest and largest prime factors of n. Find all positive integers n such that $p^2 + q^2 = n + 9$.

Solution

First eliminate the special case where $n = p = q$, which leads to $2p^2 = p + 9$. This has no integer solutions.

Rewrite the equation as $p^2 + q^2 - 9 = kpq$ for some positive integer k

Now, since $q \mid q^2$ and $q \mid kpq$, we have $q \mid p^2 - 9 = (p - 3)(p + 3)$ and, since q is prime, $q \mid p - 3$ or $q \mid p + 3$

Hence either $p = 3$ or $q \mid p + 3$

If $p = 3$ then $n = q^2$ and $3 \mid n$ so $q = 3$ and $n = 9$, which works.

If $q \mid p + 3$ and $p = 2$ then $q = 5$ and $n = 20$, which works.

If $p > 3$ then $q > \frac{p+3}{2}$ and so $q = p + 3$, which is even. So this is impossible.

Hence only solutions are $n = 9$ and $n = 20$.

An alternative approach, using modular arithmetic, observes that $p^2 \equiv 9 \pmod q$

Since q is prime, this implies that $p \equiv \pm 3 \pmod q$ and then proceeds as above.

6. Seth and Cain play a game. They take turns, and on each turn a player chooses a pair of integers from 1 to 50. One integer in the pair must be twice the other, and the players cannot choose any integers used previously. The first player who is unable to choose such a pair loses the game. If Seth starts, determine which player, if any, has a winning strategy.

Solution

Seth will win if the total number of moves is odd, and Cain if it is even.

A player must choose k and $2k$ for some $1 \leq k \leq 25$. This leads to the partition of the integers from 1 to 50 shown below. At any turn, a player must choose consecutive numbers from the same set.

$$\{1\ 2\ 4\ 8\ 16\ 32\} \quad \{5\ 10\ 20\ 40\}$$

$$\{3\ 6\ 12\ 24\ 48\}$$

$$\{7\ 14\ 28\} \quad \{9\ 18\ 36\} \quad \{11\ 22\ 44\}$$

$$\{13\ 26\} \quad \{15\ 30\} \quad \{17\ 34\} \quad \{19\ 38\} \quad \{21\ 42\} \quad \{23\ 46\} \quad \{25\ 50\}$$

The remaining twelve integers are in singleton sets and they cannot be chosen in the game.

The 10 sets consisting of two or three elements can each only be chosen once. As their number is even, this makes no difference to the result.

The five-element set containing 3 can only be used twice, so this too is irrelevant.

So if the players stick to these 11 sets, they will occupy 12 moves.

Hence the outcome of the game rests entirely on how the players use the sets. We call moves from these two sets 'interesting'.

Seth can guarantee a win if, on his first move, he chooses either (1, 2) or (16, 32) from the first interesting set. The two sets which remain after this removal have exactly the same structure, namely $\{k, 2k, 4k, 8k\}$. As soon as Cain chooses a pair from one of them, Seth mimics him by choosing the corresponding pair from the other. As a result, these two sets are exhausted in an odd number of moves, so the total number of moves is odd and Seth wins.

Note that it is not essential that Seth chooses (1, 2) or (16, 32) immediately, but it is harder to analyse the alternatives. If he is the first to choose a pair from the interesting sets, he will lose if it is anything but (1, 2) or (16, 32). If Cain is the first to choose such a pair, he will lose if it is anything but (1, 2) or (16, 32). If, however, he chooses one of these pairs, Seth can still win by being patient. He can temporise with uninteresting moves until Cain is forced to play a second interesting move once all the uninteresting moves have been used. Then Seth is back in control and can force a win.